5.4. Relation between roots and coefficients.

Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ be a polynomial of degree n with coefficients real or complex. Then $a_0 \neq 0$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation $f(x) = 0$. Then

$a_0x^n + a_1x^{n-1} + \cdots + a_n = a_0(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n)$

$= a_0[x^n - \Sigma\alpha_1 x^{n-1} + \Sigma\alpha_1\alpha_2 x^{n-2} - \ldots + (-1)^n(\alpha_1\alpha_2\ldots\alpha_n)]$, where

$\Sigma\alpha_1 =$ sum of the roots

$\Sigma\alpha_1\alpha_2 =$ sum of the products of the roots taken two at a time

$\ldots \quad \ldots \quad \ldots$

$\Sigma\alpha_1\alpha_2\ldots\alpha_r =$ sum of the products of the roots taken r at a time.

From the equality of polynomials it follows that

$$\begin{aligned} a_1 &= a_0(-\Sigma\alpha_1) \\ a_2 &= a_0(\Sigma\alpha_1\alpha_2) \\ a_3 &= a_0(-\Sigma\alpha_1\alpha_2\alpha_3) \\ \ldots & \quad \ldots \\ a_n &= a_0(-1)^n\alpha_1\alpha_2\ldots\alpha_n. \end{aligned}$$

Therefore $\Sigma\alpha_1 = -\frac{a_1}{a_0}$, $\Sigma\alpha_1\alpha_2 = \frac{a_2}{a_0}$, $\Sigma\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$, $\ldots, \alpha_1\alpha_2\ldots\alpha_n = (-1)^n\frac{a_n}{a_0}$.

Particular cases.

1. If α, β, γ be the roots of the cubic equation $a_0x^3+a_1x^2+a_2x+a_3=0$ then

$$\Sigma\alpha = -\frac{a_1}{a_0}, \quad \Sigma\alpha\beta = \frac{a_2}{a_0}, \quad \alpha\beta\gamma = -\frac{a_3}{a_0}.$$

2. If $\alpha, \beta, \gamma, \delta$ be the roots of the biquadratic equation $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$ then

$$\Sigma\alpha = -\frac{a_1}{a_0}, \quad \Sigma\alpha\beta = \frac{a_2}{a_0}, \quad \Sigma\alpha\beta\gamma = -\frac{a_3}{a_0}, \quad \alpha\beta\gamma\delta = \frac{a_4}{a_0}.$$

Worked Examples.

1. Solve the equation $2x^3 - x^2 - 18x + 9 = 0$ if two of the roots are equal in magnitude but opposite in sign.

Let the roots be α, β, γ and $\alpha = -\beta$.

$$\begin{aligned} \text{Then } \alpha+\beta+\gamma &= \tfrac{1}{2} && \ldots \text{(i)} \\ \alpha\beta+\beta\gamma+\gamma\alpha &= -9 && \ldots \text{(ii)} \\ \alpha\beta\gamma &= -\tfrac{9}{2} && \ldots \text{(iii)} \end{aligned}$$

Since $\alpha+\beta=0$, from (i) $\gamma = \frac{1}{2}$ and from (iii) $\alpha^2 = 9$.
Therefore $\alpha = \pm 3$. Hence the roots are $3, -3, \frac{1}{2}$.

2. Solve the equation $16x^4 - 64x^3 + 56x^2 + 16x - 15 = 0$ whose roots are in arithmetic progression.

Let the roots be $\alpha - 3\delta, \alpha - \delta, \alpha + \delta, \alpha + 3\delta$. Then

$$4\alpha = \frac{64}{16} \quad \ldots \text{(i)}$$

$$\{(\alpha-3\delta)+(\alpha+3\delta)\}\{(\alpha-\delta)+(\alpha+\delta)\} + (\alpha^2-9\delta^2) + (\alpha^2-\delta^2) = \frac{56}{16} \quad \ldots \text{(ii)}$$

$$\{(\alpha-3\delta)(\alpha+3\delta)\}\{(\alpha-\delta)+(\alpha+\delta)\} + (\alpha-\delta)(\alpha+\delta)\{(\alpha-3\delta)+(\alpha+3\delta)\} = -\frac{16}{16} \quad \ldots \text{(iii)}$$

$$(\alpha^2-9\delta^2)(\alpha^2-\delta^2) = -\frac{15}{16} \quad \ldots \text{(iv)}$$

From (i) $\alpha = 1$ and from (ii) $6\alpha^2 - 10\delta^2 = \frac{7}{2}$ or, $\delta = \pm\frac{1}{2}$.
Therefore the roots are $1-\frac{3}{2}, 1-\frac{1}{2}, 1+\frac{1}{2}, 1+\frac{3}{2}$, i.e., $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$.

3. Solve the equation $2x^4 - 5x^3 - 15x^2 + 10x + 8 = 0$, the roots being in geometric progression.

First we observe that if four numbers a, b, c, d be in geometric progression then $ad = bc$.

Let $\alpha, \beta, \gamma, \delta$ be the roots of the equation. Then

$$\begin{aligned}
\alpha+\beta+\gamma+\delta &= \tfrac{5}{2} & \cdots \quad & \text{(i)}\\
(\alpha+\delta)(\beta+\gamma)+\alpha\delta+\beta\gamma &= -\tfrac{15}{2} & \cdots \quad & \text{(ii)}\\
\alpha\delta(\beta+\gamma)+\beta\gamma(\alpha+\delta) &= -5 & \cdots \quad & \text{(iii)}\\
\alpha\delta.\beta\gamma &= 4 & \cdots \quad & \text{(iv)}\\
\alpha\delta &= \beta\gamma & \cdots \quad & \text{(v)}
\end{aligned}$$

From (i),(iii) and (v) we have $\alpha\delta = \beta\gamma = -2 \quad \cdots \quad$ (vi)

From (ii) $(\alpha+\delta)(\beta+\gamma) = -\frac{7}{2} \quad \cdots \quad$ (vii)

From (i) and (vii) it follows that $\alpha+\delta, \beta+\gamma$ are the roots of the equation $t^2 - \frac{5}{2}t - \frac{7}{2} = 0$. Therefore $t = -1, \frac{7}{2} \cdots \quad$ (viii)

From (vi) and (viii) it follows that one of the pairs (α,δ) and (β,γ) are the roots of the equation $x^2 + x - 2 = 0$ and the other pair are the roots of the equation $y^2 - \frac{7}{2}y - 2 = 0$.

Solving, we have $x = 1, -2; y = -\frac{1}{2}, 4$.

Hence the roots of the given equation are $-\frac{1}{2}, 1, -2, 4$.

4. If α be a multiple root of order 3 of the equation $x^4 + bx^2 + cx + d = 0, (d \neq 0)$, show that $\alpha = -\frac{8d}{3c}$.

Since $d \neq 0$, no root of the equation is 0. Let the roots be $\alpha, \alpha, \alpha, \beta$. Then $\alpha \neq 0, \beta \neq 0$.

$$\begin{aligned}
\text{We have } 3\alpha+\beta &= 0 & \cdots \quad & \text{(i)}\\
3\alpha^2+3\alpha\beta &= b & \cdots \quad & \text{(ii)}\\
\alpha^3+3\alpha^2\beta &= -c & \cdots \quad & \text{(iii)}\\
\alpha^3\beta &= d & \cdots \quad & \text{(iv)}
\end{aligned}$$

From (i) $\beta = -3\alpha$; from (iii) $8\alpha^3 = c$; from (iv) $3\alpha^4 = -d$.

Since $\alpha \neq 0, c \neq 0$, we have $\frac{3\alpha^4}{8\alpha^3} = -\frac{d}{c}$ or, $\alpha = -\frac{8d}{3c}$.

5. Find the relation among p, q, r, s so that the product of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ is unity.

Let $\alpha, \beta, \gamma, \delta$ be the roots and $\alpha\beta = 1$. Then

$$\begin{aligned}
\alpha+\beta+\gamma+\delta &= -p & \cdots \quad & \text{(i)}\\
(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta &= q & \cdots \quad & \text{(ii)}\\
\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) &= -r & \cdots \quad & \text{(iii)}\\
\alpha\beta\gamma\delta &= s & \cdots \quad & \text{(iv)}
\end{aligned}$$

From (iv) $\gamma\delta = s$ and from (iii) $(\gamma+\delta) + s(\alpha+\beta) = -r \quad \cdots \quad$ (v)

From (i) and (v) $\alpha+\beta = \frac{r-p}{1-s}$.

From (i) $\gamma+\delta = -p - \frac{r-p}{1-s} = \frac{ps-r}{1-s}$.

From (ii) $\left(\frac{r-p}{1-s}\right)\left(\frac{ps-r}{1-s}\right) + 1 + s = q$.

or, $(r-p)(ps-r) = (1-s)^2(q-s-1)$.

5.5. Symmetric functions of roots.

A function f of two or more variables is said to be a *symmetric function* if f remains unaltered by an interchange of any two of its variables.

For example, $f(x,y,z) = x^2y^2 + y^2z^2 + z^2x^2$ is a symmetric function of x, y, z. $f(x,y,z) = xy + yz$ is not symmetric in x, y, z, because f does not remain unaltered if x and y are interchanged.

A symmetric function of the roots of an equation is an expression that involves all the roots alike and the expression remains unaltered if any two of the roots be interchanged.

For example, if α, β, γ be the roots of a cubic equation, $\alpha^2 + \beta^2 + \gamma^2$ is a symmetric function of the roots, while $\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha$ is not a symmetric function.

A symmetric function which is the sum of a number of terms of the same type is represented by any one of its terms with a Σ (sigma) before it.

For example, the symmetric function $\alpha^2 + \beta^2 + \gamma^2$ is represented by $\Sigma\alpha^2$, $\alpha^2\beta\gamma + \beta^2\gamma\alpha + \gamma^2\alpha\beta$ is represented by $\Sigma\alpha^2\beta\gamma$.

For three variables α, β, γ, $\Sigma\alpha^2\beta^2$ stands for $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$, and for four variables $\alpha, \beta, \gamma, \delta$, $\Sigma\alpha^2\beta^2$ stands for $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\delta^2 + \beta^2\delta^2 + \gamma^2\delta^2$.

Worked Examples.

1. If α, β, γ be the roots of the cubic equation $x^3 + px^2 + qx + r = 0$, find the value of

(i) $\Sigma\alpha^2$, (ii) $\Sigma\alpha^2\beta$, (iii) $\Sigma\alpha^3$, (iv) $\Sigma\alpha^2\beta^2$, (v) $\Sigma\frac{1}{\alpha}$, (vi) $\Sigma\frac{1}{\alpha\beta}$, (vii) $\Sigma\frac{1}{\alpha^2}$.

Since α, β, γ are the roots, $\Sigma\alpha = -p$, $\Sigma\alpha\beta = q$, $\alpha\beta\gamma = -r$.

(i) $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = p^2 - 2q$,

(ii) $\Sigma\alpha^2\beta = \Sigma\alpha.\Sigma\alpha\beta - 3\alpha\beta\gamma = -pq + 3r$,

(iii) $\Sigma\alpha^3 = \Sigma\alpha^2.\Sigma\alpha - \Sigma\alpha^2\beta = (p^2 - 2q)(-p) - (-pq + 3r)$
$= -p^3 + 3pq - 3r$,

(iv) $\Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2\Sigma\alpha(\alpha\beta\gamma) = q^2 - 2pr$,

(v) $\Sigma\frac{1}{\alpha} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma} = -\frac{q}{r}$,

(vi) $\Sigma\frac{1}{\alpha\beta} = \frac{\alpha + \beta + \gamma}{\alpha\beta\gamma} = \frac{p}{r}$,

(vii) $\Sigma\frac{1}{\alpha^2} = (\Sigma\frac{1}{\alpha})^2 - 2\Sigma\frac{1}{\alpha\beta} = \frac{q^2}{r^2} - \frac{2p}{r} = \frac{q^2-2pr}{r^2}$.

Note. It is assumed in (v),(vi) and (vii) that none of the roots is zero, i.e., $r \neq 0$.

2. If $\alpha, \beta, \gamma, \delta$ be the roots of the biquadratic equation $x^4+px^3+qx^2+rx+s=0$, find the value of

(i) $\Sigma\alpha^2$, (ii) $\Sigma\alpha^2\beta$, (iii) $\Sigma\alpha^2\beta\gamma$, (iv) $\Sigma\alpha^2\beta^2$, (v) $\Sigma\alpha^2\beta^2\gamma^2$, (vi) $\Sigma\frac{1}{\alpha}$, (vii) $\Sigma\frac{1}{\alpha\beta}$, (viii) $\Sigma\frac{1}{\alpha^2}$.

Since α, β, γ are the roots,
$$\begin{aligned} \Sigma\alpha &= -p \\ \Sigma\alpha\beta &= q \\ \Sigma\alpha\beta\gamma &= -r \\ \alpha\beta\gamma\delta &= s. \end{aligned}$$

(i) $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = p^2 - 2q$,

(ii) $\Sigma\alpha^2\beta = \Sigma\alpha.\Sigma\alpha\beta - 3\Sigma\alpha\beta\gamma = -pq + 3r$,

(iii) $\Sigma\alpha^2\beta\gamma = \Sigma\alpha.\Sigma\alpha\beta\gamma - 4\alpha\beta\gamma\delta = pr - 4s$,

(iv) $\Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2\Sigma\alpha^2\beta\gamma - 6\alpha\beta\gamma\delta = q^2 - 2pr + 2s$,

(v) $\Sigma\alpha^2\beta^2\gamma^2 = (\Sigma\alpha\beta\gamma)^2 - 2\Sigma\alpha^2\beta^2\gamma\delta = r^2 - 2qs$,

(vi) $\Sigma\frac{1}{\alpha} = \frac{\Sigma\alpha\beta\gamma}{\alpha\beta\gamma\delta} = -\frac{r}{s}$,

(vii) $\Sigma\frac{1}{\alpha\beta} = \frac{\Sigma\alpha\beta}{\alpha\beta\gamma\delta} = \frac{q}{s}$,

(viii) $\Sigma\frac{1}{\alpha^2} = (\Sigma\frac{1}{\alpha})^2 - 2\Sigma\frac{1}{\alpha\beta} = \frac{r^2}{s^2} - \frac{q}{s} = \frac{r^2-2qs}{s}$.

Note. It is assumed in (vi) (vii) and (viii) that none of the roots is zero, i.e., $s \neq 0$.

3. If α, β, γ be the roots of the equation $x^3+qx+r=0$, find the value of (i) $\Sigma\alpha^5$, (ii) $\Sigma\frac{1}{\alpha^2-\beta\gamma}$.

(i) Since α, β, γ are the roots, $\Sigma\alpha = 0, \Sigma\alpha\beta = q, \alpha\beta\gamma = -r$.

$$\begin{aligned} \text{Also } \alpha^3 + q\alpha + r &= 0 \\ \beta^3 + q\beta + r &= 0 \\ \gamma^3 + q\gamma + r &= 0. \end{aligned}$$

$$\begin{aligned} \text{Therefore} \quad & \Sigma\alpha^3 + q\Sigma\alpha + 3r = 0 \\ \text{or,} \quad & \Sigma\alpha^3 = -3r. \end{aligned}$$

$$\begin{aligned} \text{Again, } \alpha^5 + q\alpha^3 + r\alpha^2 &= 0 \\ \beta^5 + q\beta^3 + r\beta^2 &= 0 \\ \gamma^5 + q\gamma^3 + r\gamma^2 &= 0. \end{aligned}$$

$$\begin{aligned}\text{Hence } \Sigma\alpha^5 &= -q\Sigma\alpha^3 - r\Sigma\alpha^2 \\ &= 3qr - r(-2q), \text{ since } \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta \\ &= 5qr.\end{aligned}$$

(ii)
$$\begin{aligned}\alpha^2 - \beta\gamma &= \alpha.\alpha - \beta\gamma \\ &= -\alpha(\beta+\gamma) - \beta\gamma, \text{ since } \alpha+\beta+\gamma=0 \\ &= -(\alpha\beta+\beta\gamma+\gamma\alpha) \\ &= -q.\end{aligned}$$

Therefore $\Sigma \frac{1}{\alpha^2-\beta\gamma} = -\frac{3}{q}$.

4. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation
$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0 \; (p_n \neq 0),$$
find the value of (i) $\Sigma\frac{1}{\alpha_1}$, (ii) $\Sigma\frac{\alpha_1}{\alpha_2}$, (iii) $\Sigma\frac{\alpha_1^2+\alpha_2^2}{\alpha_1\alpha_2}$.

Since $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots,
$$\Sigma\alpha_1 = -p_1, \Sigma\alpha_1\alpha_2 = p_2, \ldots, \alpha_1\alpha_2\ldots\alpha_n = (-1)^n p_n.$$

(i) $\Sigma\frac{1}{\alpha_1} = \frac{\Sigma\alpha_1\alpha_2\ldots\alpha_{n-1}}{\alpha_1\alpha_2\ldots\alpha_n} = \frac{(-1)^{n-1}p_{n-1}}{(-1)^n p_n} = -\frac{p_{n-1}}{p_n}$,

(ii) $\Sigma\frac{\alpha_1}{\alpha_2} = \Sigma\alpha_1.\Sigma\frac{1}{\alpha_1} - n = \frac{p_1p_{n-1}}{p_n} - n$,

(iii)
$$\begin{aligned}\Sigma\frac{\alpha_1^2+\alpha_2^2}{\alpha_1\alpha_2} &= \Sigma\left(\frac{\alpha_1}{\alpha_2}+\frac{\alpha_2}{\alpha_1}\right) \\ &= \alpha_1\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\cdots+\frac{1}{\alpha_n}\right) - 1 \\ &+\alpha_2\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\cdots+\frac{1}{\alpha_n}\right) - 1 \\ &+\cdots \quad \cdots \\ &+\alpha_n\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\cdots+\frac{1}{\alpha_n}\right) - 1 \\ &= \Sigma\alpha_1\Sigma\frac{1}{\alpha_1} - n = \frac{p_1p_{n-1}}{p_n} - n.\end{aligned}$$